SOLUTION OF AXISYMMETRIC PROBLEMS OF THE THEORY OF ELASTICITY WITH THE AID OF RELATIONS BETWEEN AXISYMMETRIC AND PLANE STATES OF STRESS

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A number of papers have been published on the question of finding relations between the solutions of the plane and axisymmetric problems of the theory of elasticity and the use of these relations for solving axisymmetric problems.

In [1,2], Weber has suggested the use of integral transformations for the transfer from stress functions of the plane state to stress functions of an axisymmetric state, and vice versa (in these cases the boundary conditions are transformed; Weber, together with other authors, does not indicate what the boundary conditions of the initial state should be in order to obtain the given conditions for the state to which the transfer takes place). Transfer from a plane to an axisymmetric state of a space by means of superposition is illustrated in [3], and Papkovich [4] has pointed out the analogy between the solutions of the plane and axisymmetric problems. In [5] Mossakovskii has derived a solution to the axisymmetric problem for a half-space by using analytic functions of a complex variable. Goletskii [6] has investigated the analogy between plane and axisymmetric problems for regions bounded by concentric circles and

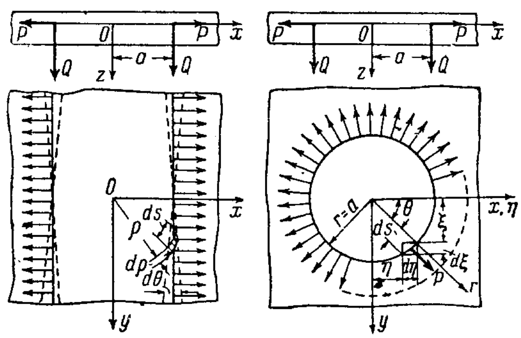

Fig. 1.

spherical surfaces, respectively. In [7,8] Polozhii considers the application of analytic functions to axisymmetric problems and establishes reversible integral transformations of axisymmetric and plane states of stress. Mustafaev [9] has investigated certain cases of transfer by means of the Weber method. Chemeris [10] has obtained an integral equation with the aid of the results of Polozhii for the axisymmetric problem with given displacements. In [11], Belen'kii gives solutions to some axisymmetric problems with the aid of integral presentations and functions of a complex variable.

In the present paper relations are derived between the plane and axisymmetric states of stress for an infinite plate, and the state of plane stress is determined which after transfer gives an axisymmetric state with known boundary conditions.

These relations enable the solution of axisymmetric problems for volumes of revolution of arbitrary shape to be reduced to the determination of two analytic functions from two integral equations [12-14].

We shall adopt the notation that all quantities referring to a state of plane stress will be distinguished by the suffix $\|$ and those referring to a state of axisymmetric stress by the suffix $^{\circ}$.

1. An infinite plate. *1. The relation between the plane and axisymmetric states derived by rotation of the plane state.* Let us suppose that an infinite plate of isotropic or transversely isotropic material with its axis of elastic symmetry parallel to the z-axis is in a state of deformation which is symmetrical with respect to the yz-plane and which is caused by the action of vertical and horizontal loads Q and P (Fig. 1, left). By rotation of the loads acting on the plate through an angle π about the z-axis, we obtain a transformed axisymmetric state.

It can be shown (for example, by replacing the loads Q and P by loads uniformly distributed over elements of area of the type shown hatched in Fig. 1, and taking into account the superposition which takes place with rotation of the loads) that in the transformed axisymmetric state loads (Figs. 2 and 3)

$$q(\rho) = \frac{2Q}{\sqrt{\rho^2 - a^2}}, \quad p(\rho) = \frac{2aP}{\rho\sqrt{\rho^2 - a^2}} \quad (\rho \geqslant a)$$
$$q(\rho) = p(\rho) = 0 \qquad (\rho < a) \tag{1.1}$$

will correspond to Q and P, which in the case of a plane state are distributed along two lines parallel to the y-axis.

In the case when $Q = Q(a)$, $P = P(a)$ for $a_0 < a < \infty$ and $Q = P = 0$ for $0 < a < a_0$, we have for $\rho \geqslant a_0$

$$q(\rho) = \int_{a_0}^{\rho} \frac{2Q(a)}{\sqrt{\rho^2 - a^2}} da,$$

$$p(\rho) = \int_{a_0}^{\rho} \frac{2aP(a)}{\rho\sqrt{\rho^2 - a^2}} da$$

for $\rho < a_0$

$$q(\rho) = p(\rho) = 0$$

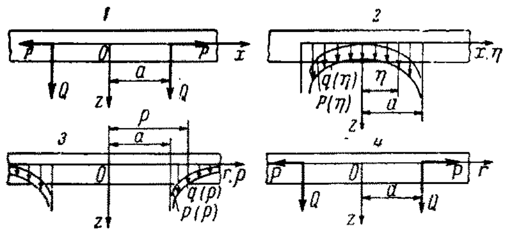

Fig. 2.

The components of the transformed axisymmetric state are given by the integrals

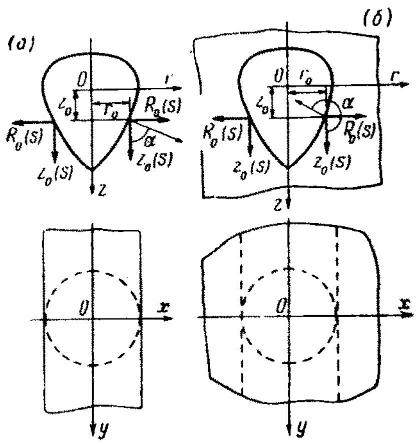

Fig. 3.

$$\sigma_{r_\circ}^* = \int_0^\pi (\sigma_{x\parallel} \cos^2\theta + \sigma_{y\parallel} \sin^2\theta)\, d\theta \tag{1.3}$$

$$\sigma_{\theta_\circ}^* = \int_0^\pi (\sigma_{y\parallel} \cos^2\theta + \sigma_{x\parallel} \sin^2\theta)\, d\theta$$

$$\sigma_{z_\circ}^* = \int_0^\pi \sigma_{z\parallel}\, d\theta$$

$$\tau_{rz^\circ}^* = \int_0^\pi \tau_{xz\parallel} \cos\theta\, d\theta$$

$$u_\circ^* = \int_0^\pi u_\parallel \cos\theta\, d\theta, \qquad w_0^* = \int_0^\pi w_\parallel\, d\theta$$

Here $\sigma_{x\parallel}(x, z)$, $\sigma_{z\parallel}(x, z)$, $\sigma_{y\parallel}(x, z)$, $\tau_{xz\parallel}(x, z)$, $w_\parallel(x, z)$, $u_\parallel(x, z)$ are the stresses and displacements for the plane state, $x = r\cos\theta$. If the plate is isotropic $\sigma_{y\parallel} = \nu(\sigma_{x\parallel} + \sigma_{z\parallel})$, and if it is transversely isotropic $\sigma_{y\parallel} = \nu_{xy}\sigma_{x\parallel} + \nu_{zx}\sigma_{z\parallel}$, ν, ν_{xy}, ν_{zx} being Poisson's ratios.

Instead of integrating with respect to θ, let us now integrate with respect to x. Relations (1.3) then become

$$\sigma_{r^\circ}^* + \sigma_{\theta^\circ}^* = \int_{-r}^{r} (\sigma_{x\parallel} + \sigma_{y\parallel}) \frac{dx}{\sqrt{r^2 - x^2}}, \qquad \sigma_{z^\circ}^* = \int_{-r}^{r} \sigma_{z\parallel} \frac{dx}{\sqrt{r^2 - x^2}}$$

$$\sigma_{r^\circ}^* - \sigma_{\theta^\circ}^* = \int_{-r}^{r} (\sigma_{x\parallel} - \sigma_{y\parallel}) \frac{2x^2 - r^2}{r^2} \frac{dx}{\sqrt{r^2 - x^2}}, \qquad \tau_{rz^\circ}^* = \int_{-r}^{r} \tau_{xz\parallel} \frac{x}{r} \frac{dx}{\sqrt{r^2 - x^2}} \tag{1,4}$$

$$u_\circ^* = \int\limits_{-r}^{r} u_\| \frac{x}{r} \frac{dx}{\sqrt{r^2 - x^2}}, \qquad w_\circ^* = \int\limits_{-r}^{r} w_\| \frac{dx}{\sqrt{r^2 - x^2}}$$

If the effect of temperature has to be taken into account, the thermal expansions for the plane and axisymmetric states are related by the expression

$$kT_\circ = \int\limits_0^\pi kT_\| d\theta = 2 \int\limits_0^r kT_\| \frac{dx}{\sqrt{r^2 - x^2}} \tag{1.5}$$

Here $T_\| (x, z)$, $T_\circ (r, z)$ are the temperatures for the plane and axisymmetric states, k is the coefficient of linear expansion, which can be constant or can depend on the temperature. Analogous relations can be derived in other cases which admit the principle of superposition (for instance, dynamic problems).

The relations derived above enable us to determine the loads (or other conditions on the surfaces of or within the plate) for the plane state, which, after transfer to the axisymmetric state by the method described, give the known loads (or other conditions) for the axisymmetric state.

If, for example, the loads for the axisymmetric state $q(\rho)$ and $p(\rho)$ are given, we can solve Equations (1.2) for the functions $Q(a)$ and $P(a)$ (by use of the substitutions $u = \rho^{-2}$ and $v = a^{-2}$ these equations become equations of the Abel type) and find the loads for the plane state, which after rotation give the known axisymmetric loads

$$Q(a) = \frac{1}{\pi} \frac{\partial}{\partial a} \int\limits_{a_\circ}^{a} q(\rho) \frac{\rho d\rho}{\sqrt{a^2 - \rho^2}}, \qquad P(a) = \frac{1}{\pi} \frac{\partial}{\partial a} \int\limits_{a_0}^{a} p(\rho) \frac{\rho^2 d\rho}{\sqrt{a^2 - \rho^2}} \quad (a_0 < a < \infty)$$

$$Q(a) = P(a) = 0 \quad (0 < a < a_0) \tag{1.6}$$

If the boundary conditions for the axisymmetric problem are given in terms of displacements, then by an analogous process we find that

$$u_\| = \frac{1}{\pi x} \frac{\partial}{\partial x} \int\limits_0^x u_\circ \frac{r^2 dr}{\sqrt{x^2 - r^2}}, \qquad w_\| = \frac{1}{\pi} \frac{\partial}{\partial x} \int\limits_0^x w_\circ \frac{r dr}{\sqrt{x^2 - r^2}} \tag{1.7}$$

In the case of a thermal expansion problem

$$kT_\| = \frac{1}{\pi} \frac{\partial}{\partial x} \int\limits_0^x kT_\circ \frac{dx}{\sqrt{x^2 - r^2}}. \tag{1.8}$$

In order to solve an axisymmetric problem it is necessary to find from (1.6) to (1.8) the loads, displacements on the boundary or other conditions of the corresponding plane state, to solve an auxiliary plane problem and to find the stresses and displacements for this state. After substitution of these stresses and displacements into (1.4) and (1.5) we can determine the stresses and displacements for the required axisymmetric state.

It should be noted that with the described method of superposition by rotation of the plane state, it is not possible to obtain any axisymmetric state, i.e. for certain forms of axisymmetric loading, Expressions (1.6) to (1.8) do not enable us to obtain a corresponding real plane state. If, for example, in the case of an axisymmetric state the loads P and Q are applied on a circle (Fig. 1, right), it is not possible to determine directly the loads of the corresponding plane state by means of Expressions (1.6).

In this case we must either make use of another superposition described in subsection 2, or we must perform certain additional operations. For example, let us consider the axisymmetric state set up by loads q_0 and p_0 uniformly distributed within the limits from $\rho = a_0$ to ∞. With the aid of Expressions (1.6) we find that in order to obtain an axisymmetric state with such loading we must rotate the plane state caused by loads

$$Q(a) = \frac{q_0}{\pi}\frac{a}{\sqrt{a^2 - a_0^2}}, \qquad P(a) = \frac{p_0}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\frac{a_0}{a} + \frac{a_0}{\sqrt{a^2 - a_0^2}}\right) \quad (a_0 < a < \infty)$$

$$Q(a) = P(a) = 0 \qquad (a_0 > a > 0) \tag{1.9}$$

Having evaluated the components of this plane state, we transfer by means of Expressions (1.4) to an axisymmetric state (set up by loads q_0 and p_0 distributed as indicated above). Differentiating with respect to a_0 the expressions for the components of the axisymmetric state so obtained, and replacing q_0 and p_0 by Q and P, we find an axisymmetric state caused by loads Q and P distributed over a circle of radius a_0.

Note that another method can be used to find an axisymmetric state with the loads P or Q acting on one circle (Fig. 1, right). Suppose that $\sigma_r^\circ(r, z, \rho)$, $\sigma_z^\circ(r, z, \rho)$, $\tau_{rz}^\circ(r, z, \rho)$, $\delta_\theta^\circ(r, z, \rho)$ are components of the required axisymmetric state, ρ is the radius of the circle on which the loads are applied. The components of the axisymmetric state produced by loads $q(\rho)$ and $p(\rho)$ distributed according to (1.1) can be expressed in the form of the integrals appearing on the right-hand sides of Expressions (1.10) and (1.11). On the other hand, the components of this state of stress $\sigma_r^{*\circ}(r, z, a)$, $\sigma^*_\theta{}^\circ(r, z, a)$, ... can be obtained by rotation of the plane state as shown in Fig. 2-1 and evaluated from Formulas (1.4). We then obtain:

when the load Q is applied

$$\sigma^*_{r^\circ}(r, z, a) = \int_a^\infty \sigma_{r^\circ}(r, z, \rho) \frac{2Q}{\sqrt{\rho^2 - a^2}} d\rho$$
$$\sigma^*_{\theta^\circ}(r, z, a) = \int_a^\infty \sigma_{\theta^\circ}(r, z, \rho) \frac{2Q}{\sqrt{\rho^2 - a^2}} d\rho \quad (1.10)$$

. . . .

and when P is applied

$$\sigma^*_{r^\circ}(r, z, a) = \int_a^\infty \sigma_{r^\circ}(r, z, \rho) \frac{2aP}{\rho\sqrt{\rho^2 - a^2}} d\rho \quad (1.11)$$
$$\sigma^*_{\theta^\circ}(r, z, a) = \int_a^\infty \sigma_{\theta^\circ}(r, z, \rho) \frac{2aP}{\rho\sqrt{\rho^2 - a^2}} d\rho$$

. . . .

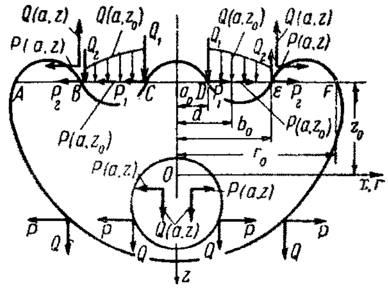

Fig. 4.

With the aid of the substitutions indicated earlier, Expressions (1.10) and (1.11) can be reduced to Abel equations, the solutions of which give:

when load Q is applied

$$\sigma_{r^\circ}(r, z, a) = \frac{2a}{\pi}\frac{\partial}{\partial a}\int_\infty^a \frac{a\sigma^*_{r^\circ}(r, z, \rho)}{\rho\sqrt{\rho^2 - a^2}} d\rho, \quad \sigma_{\theta^\circ}(r, z, a) = \frac{2a}{\pi}\frac{\partial}{\partial a}\int_\infty^a \frac{a\sigma^*_{\theta^\circ}(r, z, \rho)}{\rho\sqrt{\rho^2 - a^2}} d\rho, \ldots \quad (1.12)$$

and when P is applied

$$\sigma_{r^\circ}(r, z, a) = \frac{2a^2}{\pi}\frac{\partial}{\partial a}\int_\infty^a \frac{a\sigma^*_{r^\circ}(r, z, \rho)}{\rho^2\sqrt{\rho^2 - a^2}} d\rho, \quad \sigma_{\theta^\circ}(r, z, a) = \frac{2a^2}{\pi}\frac{\partial}{\partial a}\int_\infty^a \frac{a\sigma^*_{r^\circ}(r, z, \rho)}{\rho^2\sqrt{\rho^2 - a^2}} d\rho, \ldots \quad (1.13)$$

2. *The relation between the plane and axisymmetric states derived by linear displacement of the axisymmetric state*. Suppose that an infinite plate of isotropic or transversely isotropic material with the elastic axis parallel to the z-axis is in an axisymmetric state caused by the action of vertical and radial loads Q and P (Fig. 1, right, and Fig. 2-4). By displacing the loads acting on the plate along the y-axis from $y = -\infty$ to $y = \infty$ we obtain a certain transformed plane state.

It can be shown (for example, by replacing loads Q and P by loads uniformly distributed over elements of area of the type shown hatched in Fig. 1, and taking into account the superposition which takes place with displacement of the loads) that in the transformed plane state the loads (Fig. 2-2)

$$q(\eta) = \frac{2aQ}{\sqrt{a^2-\eta^2}}, \qquad p(\eta) = \frac{2\eta P}{\sqrt{a^2-\eta^2}} \qquad (\eta \leqslant a) \tag{1.14}$$

$$q(\eta) = p(\eta) = 0 \qquad (\eta > a)$$

will correspond to loads Q and P, which in the axisymmetric state act on one circle.

In the case when $Q = Q(a)$, $P = P(a)$ for $0 < a < c$ and $Q = P = 0$ for $\infty > a > c$

$$q(\eta) = \int_\eta^c \frac{2aQ(a)}{\sqrt{a^2-\eta^2}}\, da, \qquad p(\eta) = \int_\eta^c \frac{2\eta P(a)}{\sqrt{a^2-\eta^2}}\, da \qquad (\eta \leqslant c)$$

$$q(\eta) = p(\eta) = 0 \qquad (\eta > c) \tag{1.15}$$

The components of the transformed plane state are given by the integrals

$$\sigma_{x\parallel}{}^* = \int_{-\infty}^{\infty} (\sigma_{r^\circ}\cos^2\theta + \sigma_{\theta^\circ}\sin^2\theta)\, dy, \quad \sigma_{y\parallel}{}^* = \int_{-\infty}^{\infty} (\sigma_{r^\circ}\sin^2\theta + \sigma_{\theta^\circ}\cos^2\theta)\, dy, \tag{1.16}$$

$$\sigma_{z\parallel}{}^* = \int_{-\infty}^{\infty} \sigma_{z^\circ}\, dy, \quad \tau_{xz\parallel}{}^* = \int_{-\infty}^{\infty} \tau_{rz^\circ}\cos\theta\, dy, \quad u_\parallel{}^* = \int_{-\infty}^{\infty} u_\circ\cos\theta\, dy, \quad w_\parallel{}^* = \int_{-\infty}^{\infty} w_\circ\, dy$$

We replace the integrals with respect to y within the limits $y = -\infty$ and $y = \infty$ by double integrals with respect to y within the limits $y = 0$ and $y = \infty$, and transfer from integration with respect to y to integration with respect to r. (The expression for dy can be found by partial differentiation of the expression $x^2 + y^2 = r^2$, $\sin\theta = y/r$, $\cos Q = x/r$)

$$\sigma_{x\parallel}{}^* - \sigma_{y\parallel}{}^* = 2\int_x^\infty (\sigma_{r^\circ} - \sigma_{\theta^\circ})\, \frac{2x^2 - r^2}{r\sqrt{r^2-x^2}}\, dr, \qquad \sigma_{z\parallel}{}^* = 2\int_x^\infty \sigma_{z^\circ}\frac{r\, dr}{\sqrt{r^2-x^2}}$$

$$\sigma_{x\parallel}{}^* + \sigma_{y\parallel}{}^* = 2\int_x^\infty (\sigma_{r^\circ} + \sigma_{\theta^\circ})\, \frac{r\, dr}{\sqrt{r^2-x^2}}, \qquad \tau_{xz\parallel}{}^* = 2\int_x^\infty \tau_{rz^\circ}\frac{x\, dr}{\sqrt{r^2-x^2}}$$

$$u_\parallel{}^* = 2\int_x^\infty u_\circ\frac{x\, dr}{\sqrt{r^2-x^2}}, \qquad w_\parallel{}^* = 2\int_x^\infty w_\circ\frac{r\, dr}{\sqrt{r^2-x^2}} \tag{1.17}$$

If changes in temperature must be taken into account the thermal expansions for the axisymmetric and plane states are related by the expression

$$kT_\parallel{}^* = \int_{-\infty}^{\infty} kT_\circ\, dy = 2\int_x^\infty kT_\circ\, \frac{r\, dr}{\sqrt{r^2-x^2}} \tag{1.18}$$

Analogous relations can be obtained for other cases which admit the principle of superposition.

We will solve the first of Equations (1.17) for the components of the axisymmetric state. We set $g = r^2$, $h = x^2$, multiply both sides of the equation by $(h - H)^{-1/2}dh$, integrate them from H to B^2 and, making use of Dirichlet's formula, change the order of integration

$$\int_H^{B^2} \frac{\sigma_{x\parallel}^* - \sigma_{y\parallel}^*}{\sqrt{h-H}}\, dh = \int_H^{B^2} dh \int_h^{B^2} (\sigma_{r^\circ} - \sigma_{\theta^\circ}) \frac{2h-g}{g\sqrt{g-h}\sqrt{h-H}}\, dg$$
$$= \int_H^{B^2} \frac{\sigma_{r^\circ} - \sigma_{\theta^\circ}}{g}\, dg \int_H^{g} \frac{(2h-g)\, dh}{\sqrt{g-h}\sqrt{h-H}} = \int_H^{B^2} \pi H \frac{\sigma_{r^\circ} - \sigma_{\theta^\circ}}{g}\, dg$$

Returning now to the previous variables and differentiating both sides with respect to r, carrying out the integration by parts and letting $B \to \infty$, we find that

$$\sigma_{r^\circ} - \sigma_{\theta^\circ} = -\frac{r}{\pi}\frac{\partial}{\partial r}\left[\frac{\sigma_{x\parallel}^* - \sigma_{y\parallel}^*}{r^2}\sqrt{x^2-r^2}\right]_{r\ \ B\to\infty}^{B} - \frac{1}{\pi}\int_r^\infty \frac{\partial(\sigma_{x\parallel}^* - \sigma_{y\parallel}^*)}{\partial x}\frac{2x^2-r^2}{r^2\sqrt{x^2-r^2}}\, dx$$

The first term on the right-hand side can be expressed in the form

$$\frac{2}{\pi r^2}\lim\,[x(\sigma_{x\parallel}^* - \sigma_{y\parallel}^*)] \qquad \text{as} \quad x\to\infty$$

After solving in an analogous way the remaining equations of (1.17) and taking into account the manner in which the stresses decrease at infinity, we find that

$$\sigma_{r^\circ} - \sigma_{\theta^\circ} = -\frac{1}{\pi}\int_r^\infty \frac{\partial(\sigma_{x\parallel}^* - \sigma_{y\parallel}^*)}{\partial x}\frac{2x^2-r^2}{r^2\sqrt{x^2-r^2}}\, dx + \frac{c}{r^2}$$

$$\sigma_{z^\circ} = -\frac{1}{\pi}\int_r^\infty \frac{\partial\sigma_{z\parallel}^*}{\partial x}\frac{dx}{\sqrt{x^2-r^2}} \qquad \left(c = \frac{2}{\pi}\lim_{x\to\infty}[x(\sigma_{x\parallel}^* - \sigma_{y\parallel}^*)]\right)$$

$$\sigma_{r^\circ} + \sigma_{\theta^\circ} = -\frac{1}{\pi}\int_r^\infty \frac{\partial(\sigma_{x\parallel}^* + \sigma_{y\parallel}^*)}{\partial x}\frac{dx}{\sqrt{x^2-r^2}}, \qquad w_\circ = -\frac{1}{\pi}\int_r^\infty \frac{\partial w_\parallel^*}{\partial x}\frac{dx}{\sqrt{x^2-r^2}}$$

$$\tau_{rz^\circ} = -\frac{1}{\pi}\int_r^\infty \frac{\partial\tau_{xz\parallel}^*}{\partial x}\frac{x\, dx}{r\sqrt{x^2-r^2}}, \qquad u_\circ = -\frac{1}{\pi}\int_r^\infty \frac{\partial u_\parallel^*}{\partial x}\frac{x\, dx}{r\sqrt{x^2-r^2}} \qquad (1.19)$$

If the vector sum of the forces applied to the boundary vanishes, then $c = 0$.

Note that if we substitute in (1.17) expressions for the stresses for an axisymmetric state in terms of a stress function $\psi(r, z)$ with the aid of the expressions

$$\sigma_{r^\circ} = \frac{\partial}{\partial z}\left(\nu\nabla^2\psi - \frac{\partial^2\psi}{\partial r^2}\right), \qquad \sigma_{z^\circ} = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\psi - \frac{\partial^2\psi}{\partial z^2}\right]$$

$$\sigma_{\theta^\circ} = \frac{\partial}{\partial z}\left(\nu\nabla^2\psi - \frac{1}{r}\frac{\partial\psi}{\partial r}\right), \qquad \tau_{rz^\circ} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\psi - \frac{\partial^2\psi}{\partial z^2}\right]$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}$$

we obtain

$$\psi(r, z) = -\frac{1}{\pi}\int dz\int dz\int\left\{\frac{\partial}{\partial r} r\int_r^\infty [(2-\nu)\sigma_{x\parallel}{}^* + (1-\nu)\sigma_{z\parallel}{}^*]\frac{dx}{x\sqrt{x^2-r^2}}\right\}dz + \\ + \alpha(r)^* z^2 + \beta(r)z + \gamma(r) \tag{1.20}$$

By substituting Expressions (1.20) into (1.17) we obtain a system of integral equations for determining the functions $\alpha(r)$, $\beta(r)$, $\gamma(r)$.

In order to solve the axisymmetric problem it is necessary, with the aid of relations (1.14) to (1.18), to find the boundary conditions of the corresponding transformed plane state and, having solved the auxiliary plane problem, to find the stresses and displacements for this state of stress. We then substitute these stresses and displacements into Expressions (1.19) to find the required axisymmetric state.

2. Solution of the axisymmetric problem for a volume of revolution with the aid of analytic functions. *3. A solid body.* Let us suppose that a cylinder of isotropic or transversely isotropic material with its axis of elastic symmetry parallel to the z-axis is in a state of plane deformation which is symmetrical with respect to the yz-plane (Fig. 3a). By rotating the contour of the cross-section of the cylinder about the z-axis, we can form a volume of revolution. Initially, we make the supposition that the contour of the cross-section is such that this operation is possible, i.e. we assume that the function $r(z)$ is single-valued for the half of the contour situated on one side of the z-axis. We then superpose the states of stress and of strain for this body by rotating them through an angle π about the z-axis. The components of the transformed axisymmetric state so obtained can be found from the same relations (1.3) to (1.5) as for an infinite plate.

We introduce into (1.4) the well-known expressions for the components of the plane state in terms of two analytic functions. If the material is isotropic

$$\sigma_{x\parallel} + \sigma_{z\parallel} = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}], \qquad \sigma_{z\parallel} - \sigma_{x\parallel} + 2i\tau_{xz\parallel} = 2[\bar{\zeta}\Phi'(\zeta) + \Psi(\zeta)]$$

$$2\mu(u_\parallel + iw_\parallel) = (3-4\nu)\varphi(\zeta) - \overline{\zeta\varphi'(\zeta)} - \overline{\Psi(\zeta)} \tag{2.1}$$

Here $\zeta = x + iz$, $\bar{\zeta} = x - iz$, μ are the Lamé parameters.

We now express the analytic functions $\Phi(\zeta)$, $\Psi(\zeta)$, $\phi(\zeta)$, $\psi(\zeta)$ in terms of Cauchy integrals, change the order of integration and then integrate with respect to x. By virtue of the symmetry of the plane state about the yz-plane, it follows that

$$\Phi(t) = \overline{\Phi(-\bar{t})}, \qquad \varphi(t) = -\overline{\varphi(-\bar{t})}$$

where $t = r + iz$, $\bar{t} = r - iz$, $t_0 = r_0 + iz_0$, $\bar{t}_0 = r_0 - iz_0$, r, z, r_0, z_0 are the coordinates of points on the contour of the meridian section of the body. We make a cut in the plane of the complex variable close to the boundary of the body, and instead of carrying out the integration along its contour, we integrate along the boundaries of the cut. We note that the root $\sqrt{[(t - t_0)(t + \bar{t}_0)]}$ which appears in the integrands changes sign on passing from one side of the cut to the other.

After transformation, the values of the components of the axisymmetric state on the contour of the body can be expressed in terms of the boundary values of two analytic functions in the following way:

$$\sigma_{r_0} + \sigma_{\theta_0} = -i \int_{t_0}^{-\bar{t}_0} [2(1+2\nu)\Phi(t) - (t - t_0 + \bar{t}_0)\Phi'(t) - \Psi(t)] \frac{dt}{\sqrt{(t-t_0)(t+\bar{t}_0)}}$$

$$\sigma_{r^\circ} - \sigma_{\theta^\circ} = -\frac{i}{(t_0+\bar{t}_0)^2} \int_{t_0}^{-\bar{t}_0} [2(1-2\nu)\Phi(t) - (t - t_0 + \bar{t}_0)\Phi'(t) - \Psi(t)] \times$$
$$\times \frac{2[2t - (t_0 - \bar{t}_0)]^2 - (t_0 + \bar{t}_0)^2}{\sqrt{(t-t_0)(t+\bar{t}_0)}} dt$$

$$\sigma_{z^\circ} = -i \int_{t_0}^{-\bar{t}_0} [2\Phi(t) + (t - t_0 + \bar{t}_0)\Phi'(t) + \Psi(t)] \frac{dt}{\sqrt{(t-t_0)(t+\bar{t}_0)}}$$

$$\tau_{rz^\circ} = -\frac{1}{t_0+\bar{t}_0} \int_{t_0}^{-\bar{t}_0} [(t - t_0 + \bar{t}_0)\Phi'(t) + \Psi(t)] \frac{2t - t_0 + \bar{t}_0}{\sqrt{(t-t_0)(t+\bar{t}_0)}} dt \tag{2.2}$$

$$u_\circ = -\frac{i}{2\mu(t_0+\bar{t}_0)} \int_{t_0}^{-\bar{t}_0} [(3-4\nu)\varphi(t) - \psi(t) - (t - t_0 + \bar{t}_0)\varphi'(t)] \frac{2t - t_0 + \bar{t}_0}{\sqrt{(t-t_0)(t+\bar{t}_0)}} dt$$

$$w_\circ = -\frac{1}{2\mu} \int_{t_0}^{-\bar{t}_0} [(3-4\nu)\varphi(t) + \psi(t) + (t - t_0 + \bar{t}_0)\varphi'(t)] \frac{dt}{\sqrt{(t-t_0)(t+\bar{t}_0)}}$$

The integration is carried out over the positive branch of the root.

Note that when writing down Expressions (2.1), if we reverse the positions of the real and imaginary axes, Expressions (2.2) assume a more symmetrical (and sometimes more convenient) form: $-\bar{t}_0$ is replaced everywhere by $\bar{t}_0$.

4. *A medium containing a cavity.* Suppose that an elastic medium with an axisymmetric cavity is in a axisymmetric state of stress (Fig. 3b). By displacing the contour of the meridian section of the cavity along the y-axis from $y = -\infty$ to $y = \infty$, we form a cylindrical cavity (for this operation to be possible we initially make the same requirement of the contour of the meridian section of the cavity as that in subsection 3 for the contour of a section of a cylinder). For a medium with such a cylindrical cavity we can superpose the states of stress and of strain by displacing them along the y-axis from $y = -\infty$ to $y = \infty$. The components of the transformed plane state so obtained are given by the same expressions (1.16) to (1.19) as for an infinite plate.

The derivatives with respect to x of the components of the plane state which appear in (1.19) can be considered as components of some other plane state, and with the aid of (2.1) they can be expressed in terms of two analytic functions. Note that, since the components $\sigma_{x\parallel}, \sigma_{z\parallel}, \ldots$ characterize a plane state which is symmetrical with respect to the yz-plane, the components $\partial\sigma_{x\parallel}/\partial x, \ldots \partial\sigma_{z\parallel}/\partial x$ characterize a plane state which is skew-symmetrical about this plane. It follows that $\Phi(t) = -\overline{\Phi(-\bar{t})}$, $\phi(t) = \overline{\phi(-\bar{t})}$. After carrying out transformations analogous to those described in subsection 3, we obtain the values of the components of an axisymmetric state on the contour of a body of isotropic material expressed in terms of the boundary values of two analytic functions. If we equate to zero the vector sum of the forces applied to the contour of the cavity, these expressions coincide with (2.2).

5. *A body of arbitrary shape.* The fact that two different superpositions give analogous expressions for the components of an axisymmetric state in terms of the boundary values of analytic functions leads us to suppose that these expressions are sufficiently general, and that we can discard the requirement introduced in subsections 3 and 4 that the function $r(z)$ be single-valued for half the contour of the cross-section of the body situated on one side of the z-axis.

We will show that it is possible to make superpositions which are slightly different from those described in subsections 3 and 4, but which lead to analogous results when this requirement is removed. We shall consider a body in a state of plane deformation as part of an elastic medium subjected to loads Q and P applied on the contour $r(z)$ and loads $Q(a, z)$ and $P(a, z)$ distributed outside the meridian section of the body (Fig. 4). (Note that with the given boundary conditions the loading is many-valued.) By rotating these loads through an angle π about the z-axis, we obtain an axisymmetric state of the body. As a result of rotating the loads of the plane state Q_1, Q_2, $Q(a, z_0)$, P_1, P_2, $P(a, z_0)$

acting on the segments BC and DE of the line AF in accordance with Expressions (1.1) and (1.2) with $\rho > b_0 > a_0$, we obtain axisymmetric loads of the form

$$q(\rho, z_0) = \frac{2Q_1}{\sqrt{\rho^2 - a_0^2}} + \frac{2Q_2}{\sqrt{\rho^2 - b_0^2}} + \int_{a_0}^{b_0} \frac{2Q(a, z_0)}{\sqrt{\rho^2 - a^2}} da$$
$$p(\rho, z_0) = \frac{2a_0P_1}{\rho\sqrt{\rho^2 - a_0^2}} + \frac{2b_0P_2}{\rho\sqrt{\rho^2 - a_0^2}} + \int_{a_0}^{b_0} \frac{2aP(a, z_0)}{\rho\sqrt{\rho^2 - a^2}} da \tag{2.3}$$

For certain relations between loads Q_1, Q_2, P_1, P_2 and $Q(a, z_0)$ and $P(a, z_0)$ we can obtain $q(\rho, z_0) = p(\rho, z_0) = 0$ for $\rho > b_0 > a_0$. It follows that by rotation of the plane state of a medium we can obtain, without the conditions imposed on the contour in subsections 3 and 4, an axisymmetric state without any loads acting within the meridian section of the body.

We shall ascertain whether or not we can in this case express the relations (1.1) in the form (1.2).

Suppose that the integration of the components of the plane state (Expressions (1.4)) is carried out along the line AF, the segments BC and DE of which pass through the regions where the loads Q_1, Q_2, P_1, P_2, $Q(a, z_0)$, $P(a, z_0)$ are applied (Fig. 4). We shall consider these loads as body forces, Q_1, Q_2, P_1, P_2 being treated as loads $Q_1(a, z_0)$, $Q_2(a, z_0)$, ..., distributed along the line AF over segments of length d, so that $Q_1 = Q_1(a, z_0)d$, $Q_2 = Q_2(a, z_0)d$, ..., (later we shall let $d \to 0$). We express the components of the plane state caused by loads $Q = Q(\zeta_0)$ and $P = P(\zeta_0)$ acting on DE in the form of integrals of the stresses and displacements at the point ζ, which does not lie on the line AF, caused by concentrated forces Q and P acting at the point ζ_0 on the line AF. We must bear in mind that these stresses and displacements are given by Expressions (2.1) and by the functions

$$\Phi(\zeta) = -\frac{P + iQ}{8\pi(1-\nu)} \frac{1}{\zeta - \zeta_0}, \qquad \Psi(\zeta) = \frac{3-4\nu}{8\pi(1-\nu)} \frac{P - iQ}{\zeta - \zeta_0} - \frac{\bar{\zeta}_0(P+iQ)}{8\pi(1-\nu)} \frac{1}{(\zeta-\zeta_0)^2}$$
$$\varphi(\zeta) = -\frac{P + iQ}{8\pi(1-\nu)} \ln(\zeta - \zeta_0)$$
$$\psi(\zeta) = \frac{(3-4\nu)(P - iQ)}{8\pi(1-\nu)} \ln(\zeta - \bar{\zeta}_0) + \bar{\zeta}_0 \frac{P + iQ}{8\pi(1-\nu)} \frac{1}{\zeta - \zeta_0}$$

which are analytic everywhere except at the point $\zeta = \zeta_0$. In determining the stresses as the point ζ approaches the point ζ^* on the line AF we make use of the formulas of Sokhotskii-Plemel'

$$\lim_{\zeta \to \zeta^*} \frac{1}{2\pi i} \int_a^b \frac{f(\zeta_0)\, d\zeta_0}{\zeta_0 - \zeta} = \pm \frac{1}{2} f(\zeta^*) + \frac{1}{2\pi i} \int_a^b \frac{f(\zeta_0)\, d\zeta_0}{\zeta_0 - \zeta^*} \tag{2.4}$$

Substituting the stresses of the plane state so obtained into relations (1.1), we find the stresses for the axisymmetric state as the sum of two terms corresponding to the first and second terms of the right-hand side of Formula (2.4). If the loads of the plane state are such that for $\rho > b_0$ $q(\rho, z_0) = p(\rho, z_0) = 0$, then by comparing Expressions (1.1) and (1.5) we can see that the first terms in the expressions for σ_z°, σ_r° + σ_θ°, τ_{rz}° are zero. The corresponding term appearing in σ_r° − σ_θ° is nonzero, and in order to determine it, it is necessary to make use of the equations of equilibrium, of compatibility of strains and of uniqueness of the displacements for an axisymmetric problem. We should point out that the equations of equilibrium and of compatibility are satisfied if the required term in σ_r° − σ_θ° is of the form cr^{-2}. The integrals appearing on the right-hand sides of Expressions (2.4) and figuring here in the sense of a principal value, together with the integrals appearing in the expression for the displacements $u_{\|}$ and $w_{\|}$ can be represented by analytic functions and lead to expressions analogous to (2.2).

The condition imposed on $r(z)$ can therefore be removed.

Analogous results can be obtained (by filling up a body containing axisymmetric cavities to form an elastic medium and applying loads Q and P on the contour $r(z)$, and loads $Q(a, z)$ and $P(a, z)$ outside a meridian section of the body within the cavities) for superposition by means of a linear displacement of the axisymmetric state.

6. *The equations of the problem.* Substituting Expressions (2.2) into the boundary condition of the problem (for the first basic problem, into the relations $R_0 = \sigma_r$° $\sin a + \tau_{rz}$° $\cos a$, $Z_0 = \sigma_{r0} \cos a + \tau_{rz0} \sin a$), we obtain a set of two integral equations for the determination of the boundary values of the two analytic functions. Having solved the equations and determined the analytic functions for these boundary values we can find with the aid of Expressions (2.1) the components of an auxiliary plane state, and then with the aid of (1.4) and (1.19), we can find the components of the required axisymmetric state.

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